Soundness and Completeness

PHI 012 Lecture Notes

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Entailment and Provability

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- $\vDash$; entailment.
- $\vdash$; provability.

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- Entailment iff provability.
- Tautology iff theorem.
- Equivalent iff interderivable.
- Inconsistent iff can derive $\bot$.

And so on. Why?

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But they are connected!

- Entailment iff provability.
- Tautology iff theorem.
- Equivalent iff interderivable.
- Inconsistent iff can derive $\bot$.

And so on. Why? Because TFL is both sound and complete.

Main Terms

Soundness Defined

A proof system is sound iff there is no derivation of any semantically[1] invalid argument. Can't prove any bad arguments.

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Soundness Defined

A proof system is sound iff there is no derivation of any semantically[1] invalid argument. Can't prove any bad arguments.

Completeness Defined

A proof system is complete iff there is a derivation for every semantically valid argument. Can prove all good arguments.

[1] I.e. bad according to the truth tables.

Soundness

Soundness Defined

A proof system is sound iff there is no derivation of any semantically invalid argument. Can't prove any bad arguments.

Soundness Theorem

For any set of sentences Γ[2] *and sentence* $\mathscr{C}$*: if* $\Gamma \vdash \mathscr{C}$*, then* $\Gamma \vDash \mathscr{C}$*.*

[2] Γ is like the more familiar $\mathscr{A}_1, \ldots, \mathscr{A}_n$.

Shiny Lines

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A line n on a proof is shiny iff the assumptions on which that line depends, Δ_n, entail the sentence that appears on line n.

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A line *n* on a proof is shiny iff the assumptions on which that line depends, Δ_n, entail the sentence that appears on line *n*.

1	$F \to (G \land H)$	
2	$\quad F$	
3	$\quad G \land H$	$\to$E, 1, 2
4	$\quad G$	$\land$E, 3
5	$F \to G$	$\to$I, 2–4

Shininess Lemma and Soundness Sketch

Shininess Lemma

Every line of a TFL-proof is shiny.

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Soundness Sketch

Suppose $\Gamma \vdash \mathscr{C}$. If so, there is a proof with $\mathscr{C}$ on its last line whose only undischarged assumptions are in Γ. By the Shininess Lemma, the last line is shiny; i.e. $\Gamma \vDash \mathscr{C}$. □

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Suppose $\Gamma \vdash \mathscr{C}$. If so, there is a proof with $\mathscr{C}$ on its last line whose only undischarged assumptions are in Γ. By the Shininess Lemma, the last line is shiny; i.e. $\Gamma \vDash \mathscr{C}$. □

It remains to prove the Shininess Lemma. How?

Rule-Soundness

Rule-Sound

A rule of inference is rule-sound iff for all proofs if we obtain a line on that proof by applying that rule and every earlier line is shiny, then the new line is also shiny.

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A rule of inference is rule-sound iff for all proofs if we obtain a line on that proof by applying that rule and every earlier line is shiny, then the new line is also shiny.

To prove the Shininess Lemma, we need to show rule-soundness for all connectives. If we can prove that no application of a rule will lead us astray, we can prove the Shininess Lemma.

Sketch of Proof of Shininess Lemma

Sketch of the Proof

Consider some line n on a TFL proof. The sentence on line n must be obtained by a formal inference rule (including the rule for assumptions) which is rule-sound. This is to say that if every line before n is shiny, then so is n. By strong induction on the length of TFL proofs, every line of every TFL proof is shiny. □

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Consider some line n on a TFL proof. The sentence on line n must be obtained by a formal inference rule (including the rule for assumptions) which is rule-sound. This is to say that if every line before n is shiny, then so is n. By strong induction on the length of TFL proofs, every line of every TFL proof is shiny. □

It remains to show that, in fact, every rule is rule-sound.

Three Examples

Assumption

Introducing an assumption is rule-sound.

Conjunction Introduction

$\wedge I$ *is rule-sound.*

Conjunction Elimination

$\wedge E$ *is rule-sound.*

Assumption is Rule-Sound

Proof.

If $\mathscr{A}$ is introduced as an assumption on line n, then $\mathscr{A}$ is among Δ_n, and so $\Delta_n \vDash \mathscr{A}$. □

Conjunction Introduction is Rule-Sound

Proof.

Assume that every line before n on some TFL proof is shiny, that $\wedge$I is used on line n, and let v be any valuation that makes all of Δ_n true.

Conjunction Introduction is Rule-Sound

Proof.

Assume that every line before n on some TFL proof is shiny, that $\wedge$I is used on line n, and let v be any valuation that makes all of Δ_n true.

Note that all of Δ_i are among Δ_n. By hypothesis, line i is shiny. So any valuation that makes all of Δ_i true also makes $\mathscr{A}$ true. Since v makes all of Δ_i true, it makes $\mathscr{A}$ true too. Likewise for $\mathscr{B}$.

Conjunction Introduction is Rule-Sound

Proof.

Assume that every line before n on some TFL proof is shiny, that $\wedge$I is used on line n, and let v be any valuation that makes all of Δ_n true.

Note that all of Δ_i are among Δ_n. By hypothesis, line i is shiny. So any valuation that makes all of Δ_i true also makes $\mathscr{A}$ true. Since v makes all of Δ_i true, it makes $\mathscr{A}$ true too. Likewise for $\mathscr{B}$.

So v makes both $\mathscr{A}$ and $\mathscr{B}$ true. It follows that v makes $\mathscr{A} \wedge \mathscr{B}$ true too. So any valuation that makes all of Δ_n true also makes $\mathscr{A} \wedge \mathscr{B}$ true. That is, line n is shiny, as desired. □

Conjunction Elimination is Rule-Sound

Proof.

Assume that every line before n on some TFL proof is shiny, that $\wedge$E is used on line n, and let v be any valuation that makes all of Δ_n true.

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Assume that every line before n on some TFL proof is shiny, that $\wedge$E is used on line n, and let v be any valuation that makes all of Δ_n true.

Note that all of Δ_i are among Δ_n. By hypothesis, line i is shiny. So, any valuation that makes all of Δ_i true makes $\mathscr{A} \wedge \mathscr{B}$ true too. So, v makes $\mathscr{A} \wedge \mathscr{B}$ true, and hence makes $\mathscr{A}$ true. Likewise for $\mathscr{B}$.

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Assume that every line before n on some TFL proof is shiny, that $\wedge$E is used on line n, and let v be any valuation that makes all of Δ_n true.

Note that all of Δ_i are among Δ_n. By hypothesis, line i is shiny. So, any valuation that makes all of Δ_i true makes $\mathscr{A} \wedge \mathscr{B}$ true too. So, v makes $\mathscr{A} \wedge \mathscr{B}$ true, and hence makes $\mathscr{A}$ true. Likewise for $\mathscr{B}$.

So, $\Delta_n \vDash \mathscr{A}$ and line n is shiny, as desired. □

Completeness

Completeness Defined

A proof system is complete iff there is a derivation for every semantically valid argument. Can prove all good arguments.

Completeness

For any set of sentences Γ *and sentence* $\mathscr{C}$*: if* $\Gamma \vDash \mathscr{C}$*, then* $\Gamma \vdash \mathscr{C}$*. Equivalently, if* $\Gamma \nvdash \mathscr{C}$ *then* $\Gamma \nvDash \mathscr{C}$*, by contraposition.*

Disjunctive Normal Form

To prove this, we'll need this concept:

Disjunctive Normal Form

A sentence of TFL is in disjunctive normal form iff it:

1. contains only the connectives $\wedge, \vee, \neg$;
2. only sentence letters are in the scope of $\neg$; and,
3. only sentence letters $\wedge$, and $\neg$ are in the scope of $\vee$.

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In other words, it's a disjunction of conjunctions of sentence letters and negated sentence letters.

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2. only sentence letters are in the scope of $\neg$; and,
3. only sentence letters $\wedge$, and $\neg$ are in the scope of $\vee$.

In other words, it's a disjunction of conjunctions of sentence letters and negated sentence letters. These are in disjunctive normal form:

$$(A \wedge B) \vee (\neg A \wedge C) \qquad \neg A \vee (B \wedge C) \qquad A \wedge (B \wedge C)$$

Disjunctive Normal Form Theorem

DNF Theorem

Every sentence $\mathscr{A}$ of TFL is provably equivalent to a sentence $\mathscr{A}^*$ in disjunctive normal form.

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To prove this, we'll need the fact that provably equivalent formulas are inter-replaceable:

Disjunctive Normal Form Theorem

DNF Theorem

Every sentence $\mathscr{A}$ of TFL is provably equivalent to a sentence $\mathscr{A}^*$ in disjunctive normal form.

To prove this, we'll need the fact that provably equivalent formulas are inter-replaceable:

Replacement

If $\vdash \mathscr{A} \leftrightarrow \mathscr{B}$, then if $\mathscr{C}$ is a sentence of TFL which contains $\mathscr{A}$ as a sub-sentence, and $\mathscr{C}^*$ is just like $\mathscr{C}$ except with $\mathscr{B}$ rather than $\mathscr{A}$, then $\vdash \mathscr{C} \leftrightarrow \mathscr{C}^*$.

Replacement Example

For example, because

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Using a series of equivalences, we can give a procedure for converting a sentence of TFL into an equivalent sentence in disjunctive normal form.

Equivalences I

Conditionals and Biconditionals:

$$(\mathscr{A} \to \mathscr{B}) \leftrightarrow (\neg\mathscr{A} \lor \mathscr{B})$$

$$(\mathscr{A} \leftrightarrow \mathscr{B}) \leftrightarrow ((\mathscr{A} \land \mathscr{B}) \lor (\neg\mathscr{A} \land \neg\mathscr{B}))$$

$$(\mathscr{A} \leftrightarrow \mathscr{B}) \leftrightarrow ((\neg\mathscr{A} \lor \mathscr{B}) \land (\neg\mathscr{B} \lor \mathscr{A}))$$

Equivalences I

Conditionals and Biconditionals:

$$(\mathscr{A} \rightarrow \mathscr{B}) \leftrightarrow (\neg\mathscr{A} \vee \mathscr{B})$$
$$(\mathscr{A} \leftrightarrow \mathscr{B}) \leftrightarrow ((\mathscr{A} \wedge \mathscr{B}) \vee (\neg\mathscr{A} \wedge \neg\mathscr{B}))$$
$$(\mathscr{A} \leftrightarrow \mathscr{B}) \leftrightarrow ((\neg\mathscr{A} \vee \mathscr{B}) \wedge (\neg\mathscr{B} \vee \mathscr{A}))$$

Double Negation:

$$\neg\neg\mathscr{A} \leftrightarrow \mathscr{A}$$

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$$(\mathscr{A} \leftrightarrow \mathscr{B}) \leftrightarrow ((\neg\mathscr{A} \vee \mathscr{B}) \wedge (\neg\mathscr{B} \vee \mathscr{A}))$$

Double Negation:

$$\neg\neg\mathscr{A} \leftrightarrow \mathscr{A}$$

De Morgan's Laws:

$$\neg(\mathscr{A} \vee \mathscr{B}) \leftrightarrow (\neg\mathscr{A} \wedge \neg\mathscr{B})$$
$$\neg(\mathscr{A} \wedge \mathscr{B}) \leftrightarrow (\neg\mathscr{A} \vee \neg\mathscr{B})$$

Equivalences II

Commutativity:

$$(\mathscr{A} \vee \mathscr{B}) \leftrightarrow (\mathscr{B} \vee \mathscr{A})$$
$$(\mathscr{A} \wedge \mathscr{B}) \leftrightarrow (\mathscr{B} \wedge \mathscr{A})$$

Equivalences II

Commutativity:

$$(\mathscr{A} \vee \mathscr{B}) \leftrightarrow (\mathscr{B} \vee \mathscr{A})$$
$$(\mathscr{A} \wedge \mathscr{B}) \leftrightarrow (\mathscr{B} \wedge \mathscr{A})$$

Distributivity:

$$(\mathscr{A} \vee (\mathscr{B} \wedge \mathscr{C})) \leftrightarrow ((\mathscr{A} \vee \mathscr{B}) \wedge (\mathscr{A} \vee \mathscr{C}))$$
$$(\mathscr{A} \wedge (\mathscr{B} \vee \mathscr{C})) \leftrightarrow ((\mathscr{A} \wedge \mathscr{B}) \vee (\mathscr{A} \wedge \mathscr{C}))$$

The General Procedure

In order to convert a sentence $\mathscr{A}$ to its equivalent $\mathscr{A}^*$ in DNF, we follow this general procedure:

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3. Remove double negations.

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In order to convert a sentence $\mathscr{A}$ to its equivalent $\mathscr{A}^*$ in DNF, we follow this general procedure:

1. Replace (bi)conditionals with their equivalent.
2. Use De Morgan's Laws to move negations in front of sentence letters.
3. Remove double negations.
4. Use distributivity and commutativity to ensure that no disjunction is in the scope of a conjunction.

A Worked Example I

Consider:

$$A \to \neg((A \to C) \to C)$$

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Replace the three conditionals:

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Consider:

$$A \to \neg((A \to C) \to C)$$

Replace the three conditionals:

$$\neg A \lor \neg(\neg(\neg A \lor C) \lor C)$$

Then use the De Morgan Laws until negations are only in front of sentence letters. Starting on the inside we get:

$$\neg A \lor \neg((\neg\neg A \land \neg C) \lor C)$$

A Worked Example I

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$$A \to \neg((A \to C) \to C)$$

Replace the three conditionals:

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Then use the De Morgan Laws until negations are only in front of sentence letters. Starting on the inside we get:

$$\neg A \lor \neg((\neg\neg A \land \neg C) \lor C)$$

then once more:

$$\neg A \lor (\neg(\neg\neg A \land \neg C) \land \neg C)$$

A Worked Example I

Consider:

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Replace the three conditionals:

$$\neg A \vee \neg(\neg(\neg A \vee C) \vee C)$$

Then use the De Morgan Laws until negations are only in front of sentence letters. Starting on the inside we get:

$$\neg A \vee \neg((\neg\neg A \wedge \neg C) \vee C)$$

then once more:

$$\neg A \vee (\neg(\neg\neg A \wedge \neg C) \wedge \neg C)$$

and one last time:

$$\neg A \vee ((\neg\neg\neg A \vee \neg\neg C) \wedge \neg C)$$

A Worked Example II

Now remove the double negations:

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A Worked Example II

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We have a disjunction in the scope of a conjunction, so applying commutativity:

$$\neg A \vee (\neg C \wedge (\neg A \vee C))$$

A Worked Example II

Now remove the double negations:

$$\neg A \vee ((\neg A \vee C) \wedge \neg C)$$

We have a disjunction in the scope of a conjunction, so applying commutativity:

$$\neg A \vee (\neg C \wedge (\neg A \vee C))$$

and then distribution:

$$\neg A \vee (\neg C \wedge \neg A) \vee (\neg C \wedge C)$$

which is in disjunctive normal form.

Consistency Lemma

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For any sentence $\mathscr{A}$: if $\mathscr{A} \nvdash \bot$, then $\mathscr{A} \nvDash \bot$.

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We can now use the DNF Theorem to show that provable consistency implies semantic consistency. That is, we can show:

Consistency Lemma

For any sentence $\mathscr{A}$: if $\mathscr{A} \nvdash \bot$, then $\mathscr{A} \nvDash \bot$.

In other words, if $\mathscr{A} \nvdash \bot$, then there is some valuation v such that $v(\mathscr{A}) = T$.

Proof of Consistency Lemma I

Proof Consistency Lemma

Suppose that $\mathscr{A} \nvdash \perp$. By the DNF Theorem, $\mathscr{A}$ must be equivalent to some sentence $\mathscr{A}^*$ which is in disjunctive normal form.

Proof of Consistency Lemma I

Proof Consistency Lemma

Suppose that $\mathscr{A} \nvdash \bot$. By the DNF Theorem, $\mathscr{A}$ must be equivalent to some sentence $\mathscr{A}^*$ which is in disjunctive normal form.That means $\mathscr{A}^*$ is a sentence of the form

$$\mathscr{C}_1 \vee \ldots \vee \mathscr{C}_n$$

where each C_i is a conjunction of sentence letters and the negations of sentence letters.

Proof of Consistency Lemma II

Proof of Consistency Lemma, Continued

However, there must be some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$. Why?

Proof of Consistency Lemma II

Proof of Consistency Lemma, Continued

However, there must be some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$. Why?

1. Suppose this isn't the case: for *every* $\mathscr{C}_i$ we must have $\mathscr{C}_i \vdash \bot$.

Proof of Consistency Lemma II

Proof of Consistency Lemma, Continued

However, there must be some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$. Why?

1. Suppose this isn't the case: for *every* $\mathscr{C}_i$ we must have $\mathscr{C}_i \vdash \bot$.
2. If so, we could use each of those proofs construct a proof of $\mathscr{C}_1 \vee \ldots \vee \mathscr{C}_n \vdash \bot$ with repeated use of $\vee$E.

Proof of Consistency Lemma II

Proof of Consistency Lemma, Continued

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2. If so, we could use each of those proofs construct a proof of $\mathscr{C}_1 \vee \ldots \vee \mathscr{C}_n \vdash \bot$ with repeated use of $\vee$E.
3. If so, then $\mathscr{A}^* \vdash \bot$.

Proof of Consistency Lemma II

Proof of Consistency Lemma, Continued

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2. If so, we could use each of those proofs construct a proof of $\mathscr{C}_1 \vee \ldots \vee \mathscr{C}_n \vdash \bot$ with repeated use of $\vee$E.
3. If so, then $\mathscr{A}^* \vdash \bot$.
4. Since $\mathscr{A}^*$ is equivalent to $\mathscr{A}$, then it must also be the case that $\mathscr{A} \vdash \bot$, which is a contradiction.

Proof of Consistency Lemma II

Proof of Consistency Lemma, Continued

However, there must be some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$. Why?

1. Suppose this isn't the case: for *every* $\mathscr{C}_i$ we must have $\mathscr{C}_i \vdash \bot$.
2. If so, we could use each of those proofs construct a proof of $\mathscr{C}_1 \vee \ldots \vee \mathscr{C}_n \vdash \bot$ with repeated use of $\vee$E.
3. If so, then $\mathscr{A}^* \vdash \bot$.
4. Since $\mathscr{A}^*$ is equivalent to $\mathscr{A}$, then it must also be the case that $\mathscr{A} \vdash \bot$, which is a contradiction.
5. So, by indirect proof, there must be some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$.

Proof of Consistency Lemma III

Proof of Consistency Lemma, Continued

Since there is some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$, we can define a valuation v over sentence letters by letting $v(P) = T$ if P is a conjunct of $\mathscr{C}_i$, and letting $v(P) = F$ otherwise.

Proof of Consistency Lemma III

Proof of Consistency Lemma, Continued

Since there is some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$, we can define a valuation v over sentence letters by letting $v(P) = T$ if P is a conjunct of $\mathscr{C}_i$, and letting $v(P) = F$ otherwise.

This valuation makes every sentence letter in $\mathscr{C}_i$ true by definition, along with every negated sentence letter which appears in $\mathscr{C}_i$, so $v(\mathscr{C}_i) = T$.

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Proof of Consistency Lemma, Continued

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This valuation makes every sentence letter in $\mathscr{C}_i$ true by definition, along with every negated sentence letter which appears in $\mathscr{C}_i$, so $v(\mathscr{C}_i) = T$.

But, given that $\mathscr{A}*$ is a disjunction, one of whose disjuncts is $\mathscr{C}_i$, we also have that $v(\mathscr{A}*) = T$.

Proof of Consistency Lemma III

Proof of Consistency Lemma, Continued

Since there is some $\mathscr{C}_i$ such that $\mathscr{C}_i \nvdash \bot$, we can define a valuation v over sentence letters by letting $v(P) = T$ if P is a conjunct of $\mathscr{C}_i$, and letting $v(P) = F$ otherwise.

This valuation makes every sentence letter in $\mathscr{C}_i$ true by definition, along with every negated sentence letter which appears in $\mathscr{C}_i$, so $v(\mathscr{C}_i) = T$.

But, given that $\mathscr{A}*$ is a disjunction, one of whose disjuncts is $\mathscr{C}_i$, we also have that $v(\mathscr{A}*) = T$.

But then, given that $\mathscr{A}*$ and $\mathscr{A}$ are equivalent, it follows that $v(\mathscr{A}) = v(\mathscr{A}*) = T$, and so we must have $\mathscr{A} \nvDash \bot$, as desired. □

Proof of Completeness

Now we can put this all together to give a sketch of the completeness proof for TFL.

Proof of Completeness

Suppose that:

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \nvdash \mathscr{B}$$

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Suppose that:

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \nvdash \mathscr{B}$$

Now, given this we must also have:

$$\mathscr{A}_n, \ldots, \mathscr{A}_n, \neg\mathscr{B} \nvdash \bot$$

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Why? Because if not, we could derive *any* sentence from the premises (using indirect proof).

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Suppose that:

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Now, given this we must also have:

$$\mathscr{A}_n, \ldots, \mathscr{A}_n, \neg\mathscr{B} \nvdash \bot$$

Why? Because if not, we could derive *any* sentence from the premises (using indirect proof).Given this, we must also have:

$$\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_n \wedge \neg\mathscr{B} \nvDash \bot$$

Proof of Completeness II

Proof of Completeness, Continued

So, by the consistency lemma we must have:

$$v(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_n \wedge \neg \mathscr{B}) = T$$

Proof of Completeness II

Proof of Completeness, Continued

So, by the consistency lemma we must have:

$$v(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_n \wedge \neg \mathscr{B}) = T$$

By the truth table for $\wedge$ we have:

$$v(\mathscr{A}_1) = T, \ldots, v(\mathscr{A}_n) = T$$

Proof of Completeness II

Proof of Completeness, Continued

So, by the consistency lemma we must have:

$$v(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_n \wedge \neg \mathscr{B}) = T$$

By the truth table for $\wedge$ we have:

$$v(\mathscr{A}_1) = T, \ldots, v(\mathscr{A}_n) = T$$

as well as:

$$v(\neg \mathscr{B}) = T$$

Proof of Completeness III

Proof of Completeness, Continued

By the truth table for negation we have:

$$v(\mathscr{B}) = F$$

Proof of Completeness III

Proof of Completeness, Continued

By the truth table for negation we have:

$$v(\mathscr{B}) = F$$

So, as we have a v where $v(\mathscr{A}_i) = T$ and $v(\mathscr{B}) = F$, we have:

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \not\models \mathscr{B}$$

as desired. □

Conclusion

Since TFL is both sound and complete, there is *no* gap between provability and validity.

Something is provable *iff* it's valid!